WATER RESOURCES BULLETIN
VOL. 11, NO. 3 AMERICAN WATER RESOURCES ASSOCIATION JUNE 1975

OPTIMAL CAPACITIES OF WATER SUPPLY RESERVOIRS IN SERIES AND PARALLEL[1]

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ABSTRACT: The planning of water supply reservoirs has traditionally been based on the Rippl or sequent peak analysis which applies to the design of a single reservoir. This paper incorporates the sequent peak method as the central feature in establishing a procedure for determining the sizes of several potential reservoirs located in a system of one or more rivers. Separate algorithms are developed for sites on parallel streams and for sites on the same stream. In both cases the approach is to find the combination of reservoirs which can satisfy a given constant monthly demand at a minimum total construction cost. It is shown that both problems can be cast in the form of a dynamic programming problem. A more complex system is then a combination of reservoirs in parallel and in series. An extension is given if the monthly demand is not constant but each reservoir satisfies a constant fraction of the monthly demand. (KEY TERMS: dynamic programming; multiple reservoirs; optimal capacity; river system; sequent peak method; water supply.)

INTRODUCTION

This paper is concerned with determining the sizes of multiple water supply reservoirs which are potential sources for a single user. The problem is one which may arise in the planning and design of a large river system where the available supply can be augmented from existing and potentially new reservoirs.

The foundation of the analysis is the assumption that the problem can be handled as an optimization procedure. The purpose is to find a minimum cost solution that satisfies a given demand for water. The resulting solution is expressed in terms of which reservoirs to build or extend among a set of alternative sites and to what size they should be built. Sufficient engineering planning must have preceded this analysis so that the cost of each reservoir is known as a function of its volumetric capacity. It is also a necessary prerequisite that the river system be viewed in terms of its distinct components represented by the given reservoir sites, existing or potential.

[1]Paper No. 74084 of the *Water Resources Bulletin.* Discussions are open until December 1, 1975. Revised version of paper presented at the Eighth American Water Resources Conference, St. Louis, Missouri, October 1972.

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Although analysis of the single water supply reservoir is well documented, there are very few notions in the literature on how to size and operate multiple water supply reservoirs. One of the early works in water resource systems, Maass, *et al.*, (1962) did treat the joint operation of many reservoirs but did not concern itself with capacity decisions. More recently, Hall and Dracup (1970) focus on the allocation of total storage among reservoir sites without explicitly considering how the hydrologic interdependence is given by the topology of the river system. This will tend to ignore that the marginal unit of storage has a different effect on the total draft depending on which site it is allocated to. Buras (1972) discusses various applications of dynamic programming and linear programming to the design and operation of multistructure systems. Much of the emphasis is on the formulation of the objective function to represent some appropriate economic planning goals. Nayak and Arora (1973) developed a nonlinear programming formulation for finding the capacities in a system designed for flood control, low-flow augmentation, and recreation. The solution was achieved by piecewise linearization of all the functions.

In the term "multiple water supply reservoirs," the following cases are included:

1) Many reservoirs on the same stream.

2) Reservoirs on parallel streams.

In both cases drafts from all reservoirs are to be utilized to provide a single supply; the supply may be constant or may vary with the time of year. Streams may be branched and new inflows may occur between successive reservoirs. All these situations may be dealt with via dynamic programming, an optimization method which supplants the enumerative notions which may have preceded this paper.

The principal component of the analysis is the procedure to determine the necessary capacity of a single reservoir as a function of the draft from that reservoir. This is accomplished by means of the so-called Rippl method or the slightly more general sequent peak method. It should therefore be pointed out that the reliability of the overall procedure is determined by the reliability of the Rippl or sequent peak method and this may indeed be a limiting factor. However, these methods represent the state-of-the-art and are analytically powerful and computationally easy to implement. Although the present work rests on the computer-oriented technique of dynamic programming, it is feasible to do the computations by hand even for fairly large systems.

Since Rippl's analysis of the 1880's, engineers have known how to determine the capacity of a single water supply reservoir. Under the assumption that historical records provide a strong clue to the worst hydrologic circumstances that might again occur, Rippl showed how to calculate the reservoir storage required to meet a given feasible constant draft for water supply. Babbit and Doland (1955) showed how to extend Rippl's graphical technique to the situation in which draft requirements vary through the year. But for a sentence or two, the calculation they suggest is the method now known as the sequent peak method (Fair, Geyer and Okun, 1966). Of course, they made use of only historical records; the use of the sequent peak procedure has been recommended for use with synthetic hydrologic records as well, and wisely so. The advantage of the sequent peak procedure as stated is that it is readily adaptable to programming for the computer. However, the procedure does have some well-known shortcomings (Hall and Dracup, 1970; Buras, 1972) due to its deterministic rather than truly stochastic nature.

The sequent peak method is based on a mass diagram, i.e., a graph of cumulative volume versus time. The graphical form displays the time history of the net amount of water that would accumulate at a given location of a stream if an infinitely large reservoir were available at that site. In each time period this net volume is the difference between the inflow and the draft. Graphically, the size of the reservoir required to meet this continuous draft is given by the maximum difference between a peak and a later trough. However, the draft cannot be greater than the average inflow into the reservoir. Likewise, for every site there is an upper limit to the size of reservoir that can be built. Consequently, to any capacity of reservoir within a feasible range there corresponds a maximal withdrawal rate that can be sustained under the hydrologic record.

RESERVOIRS ON PARALLEL STREAMS

Consider n existing and potential reservoirs on n different streams such that only one site is available on each stream. A long hydrologic record is available for each stream including several successive years of low flows. The cost of expanding or building a reservoir at each of the sites is known as a function of the reservoir capacity. The problem is to determine what sites to develop and what size reservoir to build at each site such that a given demand is met. Let

$I_{i,t}$ = given inflow in month t at reservoir i.

D = monthly demand to be supplied by total draft from all reservoirs; constant through the year.

d_i = constant monthly draft from reservoir i; a decision variable.

C_i = capacity of reservoir i.

K_i = current capacity of reservoir i, hence a lower bound on C_i.

y_i = draft corresponding to K_i, hence a lower bound on d_i, i.e., the current maximum yield from reservoir i.

$f_i(C_i)$ = cost as a function of capacity for reservoir i.

Note that $K_i = 0$ if no reservoir currently exists at site i. For each reservoir, i, pick values of d_i, $y_i \leq d_i \leq D$, and for each d_i use the sequent peak method to establish the corresponding reservoir capacity, C_i. (If y_i is not known *a priori,* then allow values of d_i in the interval $0 \leq d_i \leq D$. The value of y_i will then emerge from the computations.) The resulting relationship between C_i and d_i must be kept or stored as a graph or a table. The hydrology of a site will put an upper limit on the draft that can be sustained on a continuous basis. This limit may be greater than or less than D. In either case, C_i will approach infinity as d_i approaches this limiting value. Substitute for C_i in the cost function, $f_i(C_i)$, and the new cost function, $f_i(d_i)$, now has d_i as its independent variable.

We wish to minimize the total cost of all reservoirs such that the given demand is met; hence

$$\text{minimize} \quad \sum_{i=1}^{n} f_i(d_i) \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{n} d_i = D \tag{2}$$

$$d_i \geq y_i, \text{ for all i} \tag{3}$$

This formulation describes an optimization problem where the objective function is a sum of nonlinear terms, and the constraint set is a single linear equation with lower bounded variables. Without any additional information about the behavior of the objective function, this problem can be solved as a dynamic programming problem in one of its simplest forms. The procedure requires one stage for each reservoir site, i. At each stage the decision variable is d_i and the return function is $f_i(d_i)$. The state variable will be defined from the constraint as the unfulfilled demand at stage i. Further details of this procedure are readily available in basic text books, e.g., Nemhauser (1966). The problem must be solved as a discrete one since the functions of $f_i(d_i)$ are likely to be available only in tabular form. Given the d_i's of the optimal solution, the C_i's are read from the n graphs or tables of C_i versus d_i. Note that no restrictions or qualifications have been inserted as to the behavior of the cost functions, $f_i(C_i)$, or the capacity relationships, $C_i(d_i)$. Discrete dynamic programming is a highly organized form of enumeration so it requires no particular analytical properties of the functional relationships.

The more realistic problem allows a monthly or seasonal variation in the total demand instead of being a constant, D. The procedure outlined above is then applicable only if an additional assumption is made about the operating policy of the reservoir system. This amended policy is that each reservoir must provide a constant fraction of the monthly demand regardless of its magnitude. In that way, the variable draft from any reservoir is a scaled down image of the fluctuating total demand. Hence,

x_i = fraction of the demand provided by reservoir i; a decision variable.

p_i = minimum fraction which can be supplied by an existing reservoir at site i, i.e., y_i as a fraction of the highest monthly demand.

D_t = demand in month t.

Note that y_i and p_i may both be non-zero if a continuous supply can be provided at site i without the building of a reservoir. Other variables are as defined above.

In a process similar to the one above, the cost of each reservoir will be determined as a function of its relative contribution to the demand, x_i. Reservoir i will supply x_iD_t in month t; an unknown variable quantity. Each value of x_i, $p_i \leq x_i \leq 1$, therefore fixes a schedule of drafts from the reservoir, and the required reservoir capacities can be found from the sequent peak method. This computation will also yield the value of p_i if it is not already known.

Having found the cost functions, $f_i(x_i)$, for all i, the minimum total cost is given by

$$\text{minimize} \quad \sum_{i=1}^{n} f_i(x_i) \tag{4}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_i = 1 \tag{5}$$

$$x_i \geq p_i, \text{ for all i} \tag{6}$$

This formulation is nearly identical to the first one and can likewise be handled as a dynamic programming problem.

RESERVOIRS IN SERIES

The case of reservoirs in series on the same streams can also be solved by dynamic programming. A simple procedure can be employed if there is no added inflow between the reservoirs. The hydrologic profile of the stream flow is then the same for all points on the stream. Based on this flow and the required demand, the sequent peak procedure yields a total necessary capacity of C. Let

C_i = unknown capacity of reservoir i.

K_i = capacity of existing reservoir at site i.

$f_i(C_i)$ = given cost function for reservoir i.

The problem then becomes

$$\text{minimize} \quad \sum_{i=1}^{n} f_i(C_i) \tag{7}$$

$$\text{subject to} \quad \sum_{i=1}^{n} C_i = C \tag{8}$$

$$C_i \geq K_i, \text{ for all i} \tag{9}$$

A more complex dynamic programming problem emerges if there are additional inflows between the sites. Starting at the upper end of the stream, each site (i) represents a stage in the problem. The decision variables at each stage is the total combined draft from site i plus the i-1 upstream sites. As in the previous case, sufficient hydrologic records are assumed to be known for each of the incremental inflows as well as the cost for any reservoir capacity at every site.

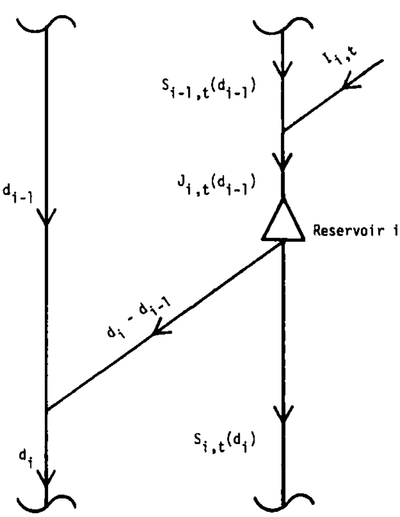

Figure 1. The i^{th} Reservoir in a Series, Showing Inflow and Outflow.

Let

D = total demand, monthly volume.

d_i = combined supply from reservoirs 1, 2, . . . i in period t.

$I_{i,t}$ = inflow added between reservoirs i-1 and i in period t.

$J_{i,t}(d_{i-1})$ = total inflow in reservoir i in period t.

$S_{i,t}(d_i)$ = spill from reservoir i in period t.

a_i = yield from existing reservoirs at sites 1, 2, . . . i.

$f_i(\cdot)$ = cost of reservoir i.

$f_i^*(d_i)$ = optimal combined cost of reservoirs 1, 2, . . . i, providing a combined supply of d_i.

Note that in accordance with these definitions it follows that

d_o = 0.

d_n = D.

$d_i - d_{i-1}$ = monthly draft from reservoir i.

a_o = 0.

$f_o^*(d_o)$ = 0.

$J_{1,t}(d_o) = I_{1,t}$.

$J_{i,t}(d_{i-1}) = I_{i,t} + S_{i-1,t}(d_{i-1})$, for $i \neq 1$.

The spill from reservoir i depends on the capacities of all the upstream reservoirs including reservoir i. However, of all the combinations of reservoir sizes which would provide exactly d_i in total draft, there is one optimal combination (a least cost solution) which will be recorded. The spill which would occur at this particular partial solution is the one noted by $S_{i,t}(d_i)$. Therefore, total inflow into the next reservoir, $J_{i+1,t}(d_i)$, is also uniquely determined by d_i.

The computational procedure is iterative, starting at the upstream end. Briefly, while site i is under consideration, the best combination of all upstream reservoirs is known in terms of the total amount of water supplied by those reservoirs. The next iteration (stage) will then generate the optimal size of reservoir i for every possible value of the combined draft when reservoir i is included. Although it will not be known how much of the demand should be satisfied by the i first reservoirs, the optimal individual releases will be known for any value of this total draft. In this fashion, each iteration corresponds to one site. The last iteration corresponds to the most downstream site where the combined draft from all reservoirs must equal the demand.

The reservoir sites are indexed from 1 to n beginning with site furthest upstream. The numbering therefore coincides with the sequence of the numerical procedure. This differs from the conventional dynamic programming algorithm which starts the computation at stage n and proceeds to stage 1. The choice of indexing does not affect the outcome or the validity of the algorithm. Furthermore, the recursive formulas often associated with a typical dynamic programming problem have been omitted here in favor of a more intuitive explanation of the procedure. Any necessary formulas are incorporated in the algorithm which is displayed in the Appendix.

There will in general be reservoirs in existence at several of the sites. It is appropriate to assume that the existing reservoirs are unable to satisfy the total demand, otherwise there would be no need to improve the system. This assumption is not strictly necessary. It will also be assumed that these reservoirs will be operated at their maximal yield. The effect of these reservoirs on the computations is to provide a lower bound on the d_i, i.e., there is a minimal draft that can be provided in the absence of any improvement program.

At site 1, the reservoir site furthest upstream, the computations are simpler than at the subsequent iterations. The object is to determine $f_1^*(d_1)$, i.e., the optimal cost of that reservoir in terms of the continuous draft of d_1. This is the same procedure as that established for the parallel reservoirs. The quantity d_1 is varied between a_1 (or 0) and D, and the necessary capacities are generated by the sequent peak procedure. Combined with $f_1(C_1)$, the result is $f_1^*(d_1)$. For every d_1 the associated spill, $S_{1,t}(d_1)$, can be computed.

At any subsequent site, i, the inflow is known since the spill from the previous site, i-1, is known. Consider any value of d_i, the total draft from site i plus all upstream reservoirs. For this value of d_i computations will be made for all possible values of draft from reservoir i, hence all possible values of d_{i-1}. Pick any d_{i-1} between a_{i-1} and d_i. The inflow can now be determined, so the cost of the reservoir which will provide $d_i - d_{i-1}$ can be found from the sequent peak procedure as before. The total combined cost is determined by adding $f_{i-1}^*(d_{i-1})$ to the cost of reservoir i. Similarly, a total cost figure can be computed for every other feasible value of d_{i-1} while keeping d_i constant. By simple comparison, the smallest of these total costs is found and is recorded as $f_i^*(d_i)$. Likewise, the associated spill, $S_{i,t}(d_i)$, is computed and recorded.

The same procedure is then carried out for every other level of d_i. If no reservoir exists at site i then a_{i-1} is a lower bound on d_i. In the event that a positive capacity does exist, then a sufficiently small value of d_i will eventually be picked such that it equals the yield from the current reservoirs down to and including reservoir i. This will be the value of a_i which will be at least as great as a_{i-1}. The total draft rates at all subsequent stages must be at least equal a_i.

The computations are carried out for all sites including the one furthest downstream. At this point it is known that $d_n = D$, i.e., the total draft must equal the demand. The ultimate value of $f_n^*(d_n)$ is the optimal total cost. The capacities of the individual reservoirs must be computed by backtracking from the solution of the final iteration, first by finding the draft rates from each reservoir. A simple table has been generated during the computations which shows the optimal value of d_{i-1} for every possible value of d_i. The individual draft rates, $d_i - d_{i-1}$, can therefore be found starting at the last site. The last phase is to determine the actual reservoir capacities. The optimal draft rates are now known, so the total inflow into each reservoir can be computed starting at the upstream site. Finally, each capacity can be found from a single application of the sequent peak analysis at each site. The details of the procedure are shown in algorithmic form in the Appendix.

The cumulative withdrawal from the reservoirs, d_i, has been conceptualized as a distinct flow separated from the stream. Alternatively, d_i can be considered to be released directly to the next downstream reservoir and the total supply, D, to be withdrawn at the last reservoir. A closer examination will reveal that these two conceptualizations result in the same analysis yielding identical solutions. A brief motivation for this reasoning is the fact that the sequent peak does not change if the inflow and draft rate are increased by identical amounts. In practical terms, however, there would be differences in the construction and operation of the two systems. Feeder lines from the reservoirs plus a separate transmission line would be required if there is a withdrawal from the stream at every reservoir. However, if the withdrawal takes place only at the downstream end of the system, then the outlet works and the stream channel must be designed to handle the higher flows.

The river may be long enough that there is a considerable time lag of the flow between the reservoirs and the confluence of the tributaries. For instance, the outflow from reservoir i-1 reaches reservoir i after a certain time period. The total inflow at i also includes some intermediate inflow into the system, e.g., a tributary. The two components of the inflow cannot be added unless there has been an adjustment for the time lag such that the two time series represent observations made at the same reference point. The choice of reference point is arbitrary since it will only shift the resulting total inflow in time and not alter the profile of the curve. Therefore, it may be advantageous to make the last site (n) the common reference point for all inflows prior to the analysis. The element of time lag will subsequently not enter into the computations.

The algorithm for reservoirs in series can also be generalized to include the case of nonconstant demand. Like the algorithm for reservoirs in parallel, it will be assumed that each reservoir provides a constant fraction of the total demand, D. Following a similar set of conversions, both the unknown decision variable and the lower bounds will be nondimensional. At the final stage, the decision variable must equal one, requiring that the complete demand be met at that point. The details of this procedure will not be shown here.

COMBINED SYSTEM

The analysis has been limited to the two special cases of water resource systems where reservoir sites are considered either in parallel or in series. The procedures differ for the two situations but at this point it can be shown that they can be combined to include a more general river system of several sites on several parallel rivers. First, each river has to be treated as one serial system whose contribution toward the demand is unknown. Therefore, it should be solved for a range in values of total draft, i.e., D must be replaced by a variable parameter in the algorithm for reservors in series. The algorithm (as given in the Appendix) must then be applied for each feasible value of this parameter. From these results are generated a cost function with respect to the total draft from that river. Similarly, a cost function can be determined for each one of the parallel streams. Finally, a dynamic programming formulation like (1), (2), (3) can be expressed directly.

CONCLUSION

The computational aspect must be addressed since it has a strong dependence on the nature of the problem. Although the modern mathematical programming ability has been developing in the wake of the growing computational potential, the present procedure can in many instances be carried out efficiently by hand. The extent of the physical problem is one determining factor as well as the required refinement of the computations, particularly in terms of the incremental values of the variables. Although conceptually the problem is one of continuous variables (draft rates, capacities), the procedure presented here is based on discrete values. The precision of the solutions increases at the expense of the computational effort when the variables are discretized with smaller intervals. In another respect, the algorithms developed here are relatively free of a type of computational complexity often encountered in dynamic programming problems. Using the terminology of dynamic programming, both the serial and parallel algorithm feature only one decision variable per stage and only one state variable.

It is important to note that this proposed analysis applies to design only. It is based on an operating rule for the individual reservoirs to provide a safe yield through a period of relative drought as defined by the hydrologic record of flows. There is a direct dependency of the method set forth here on the operating rule and it is appropriate to be aware of the inherent limitations. The flaw is the assumption that each reservoir provides either a constant draft or a constant proportion of the total draft. In reality, the operation of the system would in any one time period depend on the status of the reservoirs as well as the expected inflows. Simplistically, the draft would be provided from the fuller reservoirs or from those which expect immediate large inflows. A sophisticated operating rule would take into account the risks of the future operations based on the present situation and the hydrologic predictions. One of the features that can be added is the preferential draft from the lower reservoirs on a stream or the upper ones as the situation may present itself. Therefore, one can expect a great deal of latitude in the operation of a reservoir system far beyond the simple rule which is the basis for the sequent peak method. The literature has reflected the recent surge of interest in this particular phase of water resource management. A reservoir system as designed by the method suggested here is therefore overdesigned, i.e., there is a built-in safety factor. Depending on the desired reliability of the analysis, the output from the design phase can be tested in a simulation model where other proposed operating rules can be tried out. The compatibility between design and the operating rule is otherwise difficult to measure.

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APPENDIX

Let

D = total demand, monthly volume.

d_i = combined supply from reservoirs 1, 2, . . . i in period t.

$I_{i,t}$ = inflow added between reservoirs i-1 and i in period t.

$J_{i,t}(d_{i-1})$ = total inflow in reservoir i in period t.

$S_{i,t}(d_i)$ = spill from reservoir i in period t.

a_i = yield from existing reservoirs at sites 1, 2, . . . i.

$f_i(\cdot)$ = cost of reservoir i.

$f_i^*(d_i)$ = optimal combined cost of reservoirs 1, 2, . . . i, providing a combined supply of d_i.

Furthermore,

d_o = 0.

d_n = D.

$d_i - d_{i-1}$ = monthly draft from reservoir i.

a_o = 0.

$f_o^*(d_o)$ = 0.

$J_{1,t}(d_o)$ = $I_{1,t}$.

$J_{i,t}(d_{i-1})$ = $I_{i,t} + S_{i-1,t}(d_{i-1})$, for $i \neq 1$.

Algorithm for finding the capacities of reservoirs in series with intermediate inflows, constant demand:

Step 0
Let i = 0.
Go to step 1.

Step 1
Let i = i + 1.
Let $a_i = a_{i-1}$.
Go to step 2.

Step 2
Pick a value of d_i; $a_i \leq d_i \leq D$; starting with $d_i = D$.
If i = n, then $d_i = D$.
If $d_i = a_i$, go to step 1; otherwise, go to step 3.

Step 3
Pick a value of d_{i-1}; $a_{i-1} \leq d_{i-1} \leq d_i$; starting with $d_{i-1} = a_{i-1}$.
If i = 1, then $d_{i-1} = 0$.
If $d_{i-1} = d_i$, go to step 5; otherwise, go to step 4.

Step 4
Compute $J_{i,t}(d_{i-1})$ and $d_i - d_{i-1}$.
Find C_i from the sequent peak procedure.
Find $f_i(d_i - d_{i-1})$ from $f_i(C_i)$.
If $f_i(\cdot) = 0$, then let $a_i = d_i$.
If i = n, go to step 5; otherwise, go to step 3.

Step 5
Let $f_i^*(d_i)$ = minimum $[f_i(d_i - d_{i-1}) + f_{i-1}^*(d_{i-1})]$; $a_{i-1} \leq d_{i-1} \leq d_i$.
Record the corresponding value of d_{i-1} in a table.
Compute $S_i(d_i)$.
If i = n, go to step 6; otherwise, if $d_i = D$, go to step 1; otherwise, go to step 2.

Step 6
Given the final solution, $d_n = D$, backtrack to find d_i's: given the optimal d_i, find the corresponding d_{i-1} from the stored table; i = n, n 1, . . . , 2.
Recalculate $d_i - d_{i-1}$ and $J_{i,t}(d_{i-1})$; find C_i from the sequent peak procedure; i = 1, 2, . . .,n.